Arbitrage-free Term Structure Models of Interest Rates in the multiple-curve framework

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Introduction

In practice, we have a LIBOR curve and several discount curves in a single currency. The below table shows market rates on Oct 3, 2015. Roughly speaking, yen IRS rates corresponds yen LIBOR curve, IRS-OIS spreads corresponds to yen LIBOR-OIS curve, and USD/JPY cross currency basis swap rates corresponds to the spread curve of LIBOR and discount rates for the yen cashflows collateralized in USD cash.

	yen IRS(%)	IRS-OIS spread(bp)	usd/yen basis(bp)
1y	0.115	2.625	-50.75
2y	0.106	3	-60.5
3y	0.116	3.375	-69
5y	0.191	4.25	-80
7y	0.301	5.517	-82
10y	0.493	7.4	-75.5
20y	1.116	9.665	-53.875
30y	1.341	10.55	-44.125

We need term structure models where LIBOR curve is the main interest rate driver and LIBOR discount spread curves are additional stochastic drivers.

Compared with discount curves, LIBOR curve is "fictious" because LIBOR bond prices are not observable. Therefore it has been common to apply term-structure modeling to discount curve(e.g., OIS curve with constant LIBOR-discount spreads).

In this presentaton, we apply term-structure modeling to LIBOR curve directly and to additional several LIBOR-discount spread curves.

Dividend,numeraire and risk-neutral measure

$X(t)$: time-t FX spot rate of one unit of foreign currency in domestic currency at time t

$X(t;T)$: time-t FX forward rate settled at time T

$r(t)$: domestic ON rate at time t, $r^*(t)$: foreign ON rate

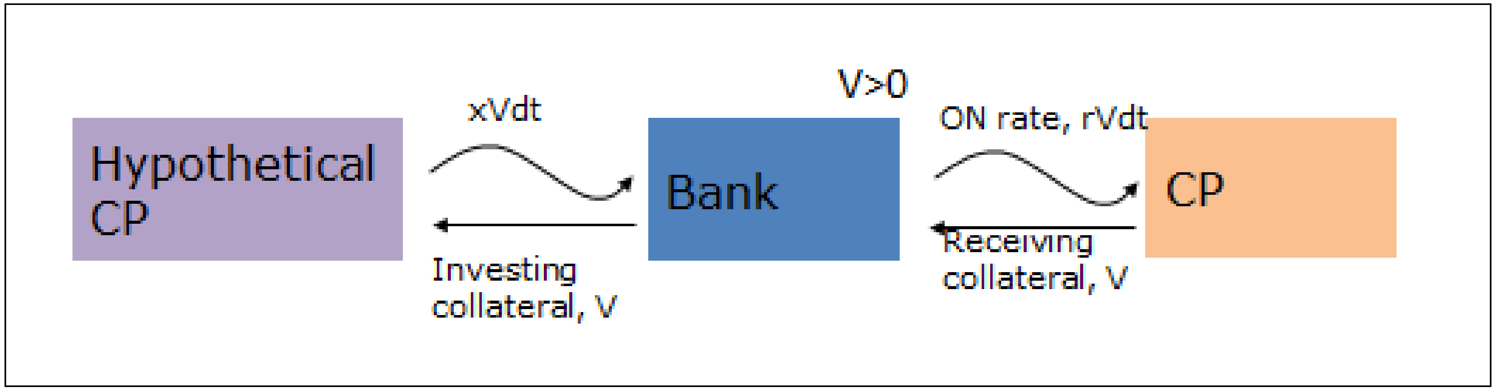
Derivatives with payoff $f(S_T)$ collateralized by domestc cash at time t:

$$
\begin{aligned}
V(t;T) &= E_t^{\mathbf{Q}}\left[e^{-\int_t^T x(u)du} f(S_T) + \int_t^T e^{-\int_t^T x(u)du}\left(x(u) - r(u)\right) V(u;T)du\right] \\
&= E_t^{\mathbf{Q}}\left[e^{-\int_t^T r(u)du} f(S_T)\right]
\end{aligned}
$$

Nummeraire can be anything:

$$B_x(t) = e^{\int_0^t x(u)du}$$

Fujii, Shimada and Takahashi(2009) considered the case where x is "risk-free rate" while Piterbarg(2010) considered it where x is the bank's funding rate.



Ignoring the "dividend" cashflows, we can think of the effective numeraire as

$$B_r(t) = e^{\int_0^t r(u)du}$$

Derivatives with payoff $f(S_T)$ collateralized by foreign cash at time t

$$\begin{aligned}\overline{V}(t;T) &= E_t^{\mathbf{Q}}\left[e^{-\int_t^T x(u)du} f(S_T) + \int_t^T e^{-\int_t^T x(u)} \left(x(u) - \overline{r}(u)\right)\overline{V}(u;T)du\right] \\ &= E_t^{\mathbf{Q}}\left[e^{-\int_t^T \overline{r}(u)du} f(S_T)\right]\end{aligned}$$

where $\overline{r}(t)$ satisfies

$$X(t)\left(1+\overline{r}(t)dt\right) = X(t;t+dt)\left(1+r^*(t)dt\right)$$

Ignoring the "dividend" cashflows, the effective numeraire is

$$B_{\overline{r}}(t) = e^{\int_0^t \overline{r}(u)du}$$

The above argument shows that the risk-neutral measure stays the same even though the effective numeraires are different.

Multiple curves

OIS curve (domestic cash-collateralized discounting curve) at time t:

$$D(t;T) = E_t^{\mathbf{Q}}\left[e^{-\int_t^T r(u)du}\right] = e^{-\int_t^T f(t;u)du}, \ ^{\forall}T \geq t$$

LIBOR curve at time t:

$$\widetilde{D}(t;T) = E_t^{\mathbf{Q}}\left[e^{-\int_t^T \widetilde{r}(u)du}\right] = e^{-\int_t^T \widetilde{f}(t;u)du}, \ ^{\forall}T \geq t$$

Foreign cash-collateralized discounting curve at time t:

$$\overline{D}(t;T) = E_t^{\mathbf{Q}}\left[e^{-\int_t^T \overline{r}(u)du}\right] = e^{-\int_t^T \overline{f}(t;u)du}, \ ^{\forall}T \geq t$$

Funding curve at time t:

$$D^F(t;T) = E_t^{\mathbf{Q}}\left[e^{-\int_t^T r^F(u)du}\right] = e^{-\int_t^T f^F(t;u)du}, \ \forall T \geq t$$

Spread curve (OIS-curve based)

LIBOR-OIS spread: Define spot and forward LIBOR-OIS spreads as

$$s(t) = \widetilde{r}(t) - r(t)$$
$$s(t;T) = \widetilde{f}(t;T) - f(t;T)$$

Define LIBOR-OIS spread DFs as

$$D_s(t;T) \equiv \frac{\widetilde{D}(t;T)}{D(t;T)}$$

then

$$D_s(t;T) = E_t^{\mathbf{F(T)}}\left[e^{-\int_t^T s(u)du}\right] = e^{-\int_t^T s(t;u)du}$$

where the OIS (riskless) T-forward measure $\mathbf{F(T)}$ is characterized from $\mathbf{Q}$ as

$$\left.\frac{d\mathbf{F(T)}}{d\mathbf{Q}}\right|_t = \frac{D(t;T)}{e^{\int_0^t r(u)du} D(0;T)}, \ 0 \le t \le T.$$

Spread curve (Libor-curve based)

OIS curve also can be written as:

$$\begin{aligned} D(t;T) &= E_t^{\mathbf{Q}}\left[e^{-\int_t^T(\widetilde{r}(u)-s(u))du}\right] \\ &= \widetilde{D}(t;T)E_t^{\widetilde{\mathbf{F}}(\mathbf{T})}\left[e^{\int_t^T s(u)du}\right] \end{aligned}$$

then another representation of LIBOR-OIS curve is:

$$D_s(t;T)^{-1} = E_t^{\widetilde{\mathbf{F}}(\mathbf{T})}\left[e^{\int_t^T s(u)du}\right] = e^{\int_t^T s(t;u)du}$$

We introduced another forward measure $\widetilde{\mathbf{F}}(\mathbf{T})$ whose numeraire is LIBOR (risky) discount bond $\widetilde{D}(\cdot;T)$. The LIBOR (risky) T-forward measure $\widetilde{\mathbf{F}}(\mathbf{T})$ is characterized from $\mathbf{Q}$ as

$$\left.\frac{d\widetilde{\mathbf{F}}(\mathbf{T})}{d\mathbf{Q}}\right|_t = \frac{e^{-\int_0^t s(u)du}\widetilde{D}(t;T)}{e^{\int_0^t r(u)du}\widetilde{D}(0;T)},\ 0 \leq t \leq T.$$

Note that $e^{-\int_0^t s(u)du}$ is needed for the density to become Q-martingale. This additional term is explained by "foreign-currency analogy".

Foreign-currency analogy

Bianchetti(2009) originally introduced "foreign-currency analogy" to resolve the double rate system in one currency. Let $\mathbf{Q}_d$ denote the domestic risk neutral measure and $\mathbf{Q}_f$ denote the foreign risk neutral measure.

$$\begin{aligned} D(t;T) &= E_t^{\mathbf{Q}_d}\left[e^{-\int_t^T r(u)du}\right] \\ \widetilde{D}(t;T) &= E_t^{\mathbf{Q}_f}\left[e^{-\int_t^T (r(u)+s(u))du}\right] \\ X(t;T)D(t;T) &= X(t)\widetilde{D}(t;T) \end{aligned}$$

where $X(t)$ is the time-t "spot FX rate" and $X(t;T)$ is the "T-matured forward FX rate". Foreign discount bond can also be written as:

$$\widetilde{D}(t;T) = \frac{1}{X(t)} E_t^{\mathbf{Q}_d}\left[e^{-\int_t^T r(u)du} X(T)\right]$$

Comparing this with LIBOR-OIS spread representations, we identify that $\mathbf{Q}_d = \mathbf{Q}_f$ and

$$\begin{aligned} X(t) &= e^{-\int_0^t s(u)du} \\ X(t;T) &= D_s(t;T) e^{-\int_0^t s(u)du} = E_t^{\mathbf{F(T)}}\left[e^{-\int_0^T s(u)du}\right] \end{aligned}$$

Indeed, since "spot FX rate" is finite variation process, there should be no difference between $\mathbf{Q}_d$ and $\mathbf{Q}_f$.

In our opinion, foreign-currency analogy is still important. Within this framwork, we can interpret LIBOR forward measure $\widetilde{\mathbf{F}}(\mathbf{T})$ as "foreign" forward measure.

Kijima, Tanaka and Wong (2009): They change measures from $\mathbf{Q}_f$ to $\mathbf{Q}_d$ using "market price of risk" in the foreign bond formula, resulting in the derivatives pricing formulas dependent on "market price of risk".

Kenyon (2010): "Spot FX rate" has a volatility. Maybe this should be a "forward FX rate"?

OIS-curve based model

For simplicity, we consider 2 scaler Brownian motions, $W_r(t)$ and $W_s(t)$, which drives OIS curve and LIBOR-OIS spread curve respectively.

Ito's lemma gives (spread) discount bond dynamics

$$\frac{dD(t;T)}{D(t;T)} = r(t)dt - b_r(t;T)dW_r^{\mathbf{Q}}(t)$$

$$\frac{dD_s(t;T)}{D_s(t;T)} = s(t)dt - b_s(t;T)dW_s^{\mathbf{F(T)}}(t)$$

where bond volatility $b(t;T)$ in general is stochastic and linked to the forward rate (spread) volatilities as

$$b_i(t;T) = \sigma_i(t;T)\int_t^T \sigma_i(u;T)du,\ i = r, s$$

We allow correlation between discount curve and spread curve;

$$dW_r(t)dW_s(t) = \rho_{r,s}(t)dt$$

Since $\widetilde{f}(t;T)$ is $\widetilde{F}(\mathbf{T})$-martingale,

$$\begin{aligned} d\widetilde{f}(t;T) &= df(t;T) + ds(t;T) \\ &= \sigma_r(t;T)dW_r^{\widetilde{\mathbf{F}}(\mathbf{T})} + \sigma_s(t;T)dW_s^{\widetilde{\mathbf{F}}(\mathbf{T})} \end{aligned}$$

On the other hand, since $f(t;T)$ is $F(T)$-martingale,

$$d\widetilde{f}(t;T) = \sigma_r(t;T)dW_r^{\mathbf{F(T)}} + ds(t;T)$$

Thus,

$$\begin{aligned} ds(t;T) &= (\sigma_r(t;T)b_s(t;T) + \sigma_s(t;T)b_r(t;T))\,\rho_{r,s}(t) \\ &\quad + \sigma_s(t;T)b_s(t;T)dt + \sigma_s(t;T)dW_s^{\mathbf{Q}} \end{aligned}$$

LIBOR-curve based model

Alternatively, we model LIBOR curve directly as:

$$\begin{aligned}
\frac{d\widetilde{D}(t;T)}{\widetilde{D}(t;T)} &= \widetilde{r}(t)dt - b_{\widetilde{r}}(t;T)dW_{\widetilde{r}}^{\mathbf{Q}}(t) \\
\frac{dD_s(t;T)^{-1}}{D_s(t;T)^{-1}} &= -s(t)dt + b_s(t;T)dW_s^{\widetilde{\mathbf{F}}(\mathbf{T})}(t) \\
dW_{\widetilde{r}}(t)dW_s(t) &= \rho_{\widetilde{r},s}(t)dt
\end{aligned}$$

Since $f(t;T)$ is $\mathbf{F}(\mathbf{T})$-martingale,

$$\begin{aligned} df(t;T) &= d\widetilde{f}(t;T) - ds(t;T) \\ &= \sigma_{\widetilde{r}}(t;T)dW_{\widetilde{r}}^{\mathbf{F(T)}} - \sigma_s(t;T)dW_s^{\mathbf{F(T)}} \end{aligned}$$

On the other hand, since $\overline{f}(t;T)$ is $\overline{\mathbf{F}}(\mathbf{T})$-martingale

$$df(t;T) = \sigma_{\widetilde{r}}(t;T)dW_{\widetilde{r}}^{\widetilde{\mathbf{F}}\mathbf{(T)}} - ds(t;T)$$

Thus,

$$\begin{aligned} -ds(t;T) &= \left(\sigma_{\widetilde{r}}(t;T)b_s(t;T) - \sigma_s(t;T)b_{\widetilde{r}}(t;T)\right)\rho_{\widetilde{r},s}(t) \\ &\quad + \sigma_s(t;T)b_s(t;T)dt - \sigma_s(t;T)dW_s^{\mathbf{Q}} \end{aligned}$$

Forward LIBORs under the LIBOR discounting

$L(T; T+\delta)$: Spot LIBOR over $[T, T+\delta]$ at time T

$$L(T; T+\delta) = \frac{1}{\delta}\left(\frac{1}{\widetilde{D}(T; T+\delta)} - 1\right)$$

$\widetilde{L}(t; T, T+\delta)$: T-maturing forward LIBOR at time t ($\leq T$) under the LIBOR discounting

$$\begin{aligned}\widetilde{L}(t; T, T+\delta) &= \frac{E_t^{\mathbf{Q}}\left(e^{-\int_t^{T+\delta}\widetilde{r}(u)du} L(T; T+\delta)\right)}{\widetilde{D}(t; T+\delta)} \\ &= E_t^{\widetilde{\mathbf{F}}(\mathbf{T}+\boldsymbol{\delta})}\left(L(T; T, T+\delta)\right)\end{aligned}$$

which shows $\widetilde{L}(t; T, T+\delta)$ is a $\widetilde{\mathbf{F}}(\mathbf{T}+\boldsymbol{\delta})$-martingale, where $\widetilde{\mathbf{F}}(\mathbf{T})$ is the LIBOR-risky T-forward measure whose numeraire is $\widetilde{D}(\cdot; T)$.

Then,

$$\widetilde{L}(t; T, T+\delta) = \frac{1}{\delta}\left(\frac{\widetilde{D}(t; T)}{\widetilde{D}(t; T+\delta)} - 1\right),\ t \leq T$$

In the context of foreign-currency analogy, all these are derived in the "foreign currency".

Collateralized forward LIBORs

$L(t;T,T+\delta)$: T-maturing forward LIBOR over $[T,T+\delta]$ at time t under the OIS discounting

$$
\begin{aligned}
L(t;T,T+\delta) &= \frac{E_t^{\mathbf{Q}}\left(e^{-\int_t^{T+\delta} r(u)du} L(T;T,T+\delta)\right)}{D(t;T+\delta)} \\
&= E_t^{\mathbf{F(T+\boldsymbol{\delta})}}\left(L(T;T,T+\delta)\right)
\end{aligned}
$$

which shows $L(t;T,T+\delta)$ is a $\mathbf{F}(\mathbf{T}+\boldsymbol{\delta})$-martingale. In the context of "foreign-currency analogy", the forward LIBOR is considered as a "quanto" rate because the LIBOR which is denominated in the foreign currency is paid in the domestic currency.

Define a positive $\widetilde{\mathbf{F}}(T)$-martingale density process M_s such that

$$
\begin{aligned}
\frac{d\mathbf{F(T)}}{d\widetilde{\mathbf{F}}\mathbf{(T)}} &= M_s(T) \\
M_s(t) &= E_t^{\widetilde{\mathbf{F}}\mathbf{(T)}}\left(M_s(T)\right),\ t \leq T \\
M_s(0) &= 1
\end{aligned}
$$

then

$$
L(t;T,T+\delta) = E_t^{\widetilde{\mathbf{F}}\mathbf{(T+\delta)}}\left(\frac{M_s(T+\delta)}{M_s(t)}L(T;T,T+\delta)\right)
$$

Changing measures to the LIBOR-risky T-forward measure $\widetilde{\mathbf{F}}(T+\delta)$

$$L(t;T,T+\delta) = \frac{\widetilde{D}(t;T+\delta)E_t^{\widetilde{\mathbf{F}}(\mathbf{T}+\boldsymbol{\delta})}\left(e^{\int_t^{T+\delta} s(u)du} L(T;T,T+\delta)\right)}{D(t;T+\delta)}$$

The density process M_s is identified as

$$M_s(t;T) \triangleq \frac{D_s(0;T)}{D_s(t;T)} e^{\int_0^t s(u)du} = D_s(0;T)E_t^{\widetilde{\mathbf{F}}(\mathbf{T})}\left(e^{\int_0^T s(u)du}\right)$$

So M_s is given by

$$M_s(t;T) \equiv \left.\frac{d\mathbf{F}(\mathbf{T})}{d\widetilde{\mathbf{F}}(\mathbf{T})}\right|_t = \exp\left(-\frac{1}{2}\int_0^t b_s(u,T)^2 du + \int_0^t b_s(u,T)dW_s^{\mathbf{F}(\mathbf{T})}(u)\right)$$

Hence, we have by Girsanov

$$dW_i^{\widetilde{\mathbf{F}}(\mathbf{T})}(t) = dW_i^{\mathbf{F}(\mathbf{T})}(t) + \rho_{i,s}(t)b_s(t,T)dt, \ i = r, s, \cdots$$

Considering $M(\cdot, T+\delta)$ is a $\widetilde{\mathbf{F}}(T+\delta)$-martingale, its dynamics can be written as

$$\frac{dM_s(t;T+\delta)}{M_s(t;T+\delta)} = b_s(t,T+\delta)dW_s^{\widetilde{\mathbf{F}}(\mathbf{T}+\boldsymbol{\delta})}(t)$$

OIS-curve based approach

$$\frac{d\left(\widetilde{D}(t;T)\Big/\widetilde{D}(t;T+\delta)\right)}{\widetilde{D}(t;T)\Big/\widetilde{D}(t;T+\delta)} = (\cdots)\,dt + \frac{d\widetilde{D}(t;T)}{\widetilde{D}(t;T)} - \frac{d\widetilde{D}(t;T+\delta)}{\widetilde{D}(t;T+\delta)}$$

$$= \left(b_r(t,T+\delta) - b_r(t,T)\right) dW_r^{\widetilde{\mathbf{F}}(\mathbf{T}+\boldsymbol{\delta})}(t)$$

$$+ \left(b_s(t,T+\delta) - b_s(t,T)\right) dW_s^{\widetilde{\mathbf{F}}(\mathbf{T}+\boldsymbol{\delta})}(t)$$

we only have an interest in the drift term of the

$$d\left(\frac{M_s(t;T+\delta)\widetilde{D}(t;T)}{\widetilde{D}(t;T+\delta)}\right)\Bigg/\frac{M_s(t;T+\delta)\widetilde{D}(t;T)}{\widetilde{D}(t;T+\delta)}$$

: i.e.,

$$
\begin{aligned}
& E_t^{\widetilde{\mathbf{F}}(\mathbf{T}+\boldsymbol{\delta})}\left[\frac{d\left(\frac{M_s(t;T+\delta)\widetilde{D}(t;T)}{\widetilde{D}(t;T+\delta)}\right)}{\frac{M_s(t;T+\delta)\widetilde{D}(t;T)}{\widetilde{D}(t;T+\delta)}}\right] \\
= \; & \left(b_r(t,T+\delta)-b_r(t,T)\right)\rho_{r,s}b_s(t,T+\delta)dt \\
& +\left(b_s(t,T+\delta)-b_s(t,T)\right)b_s(t,T+\delta)dt
\end{aligned}
$$

Thus, the collateralized forward LIBOR can be written as

$$
\begin{aligned}
L(t;T,T+\delta) &= E_t^{\widetilde{\mathbf{F}}(\mathbf{T}+\boldsymbol{\delta})}\left(\frac{M_s(T+\delta;T+\delta)}{M_s(T;T+\delta)}L(T;T+\delta)\right) \\
&= \frac{1}{\delta}E_t^{\widetilde{\mathbf{F}}(\mathbf{T}+\boldsymbol{\delta})}\left(\frac{M_s(T+\delta;T+\delta)}{M_s(T;T+\delta)}\left(\frac{1}{\widetilde{D}(T;T+\delta)}-1\right)\right) \\
&= \frac{1}{\delta}\left(\frac{\widetilde{D}(t;T)}{\widetilde{D}(t;T+\delta)}E_t^{\widetilde{\mathbf{F}}(\mathbf{T}+\boldsymbol{\delta})}\left[\exp\left(\int_t^T k(u,T)du\right)\right]-1\right)
\end{aligned}
$$

where

$$
\begin{aligned}
k(t,T) \quad = \quad & \left(b_r(t,T+\delta)-b_r(t,T)\right)\rho_{r,s}(t)b_s(t,T+\delta) \\
& +\left(b_s(t,T+\delta)-b_s(t,T)\right)b_s(t,T+\delta)
\end{aligned}
$$

In other word,

$$\frac{1+\delta L(t;T,T+\delta)}{1+\delta \widetilde{L}(t;T,T+\delta)} = E_t^{\widetilde{\mathbf{F}}(\mathbf{T}+\boldsymbol{\delta})}\left[\exp(\int_t^T k(u,T)du)\right]$$

Thus from this results, we know that only when the LIBOR-OIS spread is not stochastic, i.e., $b_s(t;T) = 0 \ \forall t, T$, then

$$L(t;T,T+\delta) = \frac{1}{\delta}\left(\frac{\widetilde{D}(t;T)}{\widetilde{D}(t;T+\delta)} - 1\right)$$

LIBOR-curve based approach

When we use the LIBOR-curve based model,

$$\frac{d\left(\widetilde{D}(t;T)\Big/\widetilde{D}(t;T+\delta)\right)}{\widetilde{D}(t;T)\Big/\widetilde{D}(t;T+\delta)} = (\cdots)\,dt + \frac{d\widetilde{D}(t;T)}{\widetilde{D}(t;T)} - \frac{d\widetilde{D}(t;T+\delta)}{\widetilde{D}(t;T+\delta)}$$

$$= \left(b_{\widetilde{r}}(t,T+\delta) - b_{\widetilde{r}}(t,T)\right) dW_{\widetilde{r}}^{\widetilde{\mathbf{F}}(\mathbf{T}+\boldsymbol{\delta})}(t)$$

Thus,

$$L(t;T,T+\delta) = \frac{1}{\delta}\left(\frac{\widetilde{D}(t;T)}{\widetilde{D}(t;T+\delta)} E_t^{\widetilde{\mathbf{F}}(\mathbf{T}+\boldsymbol{\delta})}\left[\exp\left(\int_t^T k'(u,T)du\right)\right] - 1\right)$$

where

$$k'(t,T) = \left(b_{\widetilde{r}}(t,T+\delta) - b_{\widetilde{r}}(t,T)\right)\rho_{\widetilde{r},s}(t) b_s(t,T+\delta)$$

There are 2 cases where the convexity terms disappear. One is when the LIBOR-OIS spread is not stochastic and the other is when LIBOR curve is uncorrelated with the spread curve, i.e., $\rho_{\widetilde{r},s}(t) = 0,\ \forall t$.

When we use LIBOR-curve based model, it is sufficient to assume the independence between LIBOR curve and spread curve to get rid of the convexity adjustment terms (or quanto terms in terms of foreign-currency analogy) from the collateralized forward LIBOR formula.

Initial forward LIBORs

At time 0, we have

$$L(0;T,T+\delta)=\frac{1}{\delta}\left(\frac{\widetilde{D}(0;T)}{\widetilde{D}(0;T+\delta)}E^{\widetilde{\mathbf{F}}(\mathbf{T}+\boldsymbol{\delta})}\left[\exp\left(\int_0^T k'(u,T)du\right)\right]-1\right)$$

By curve stripping, we know the initial forward LIBORs and they can be parametarized so that

$$L(0;T,T+\delta)=\frac{1}{\delta}\left(\frac{\widetilde{D}'(0;T)}{\widetilde{D}'(0;T+\delta)}-1\right)$$

holds. Some say that option market is needed to price collateralized linear trades (e.g., swaps), but it is wrong. We can just interpolate $\{\widetilde{D}'(0;T)\}_{\dot{T}>0}$ to price any linear trades statically.

Spread curve of the discount curve in the case of foreign-cash collateral

OIS-curve based

Define spot and forward "collateral currency" spreads as

$$z(t) = \overline{r}(t) - r(t)$$
$$z(t;T) = \overline{f}(t;T) - f(t;T)$$

Define "collateral currency spread" DFs as

$$D_z(t;T) \equiv \frac{\overline{D}(t;T)}{D(t;T)}$$

then

$$D_z(t;T) = E_t^{\mathbf{F(T)}}\left[e^{-\int_t^T z(u)du}\right] = e^{-\int_t^T z(t;u)du}$$

LIBOR-curve based

Alternatively, define the spread from the LIBOR curve:

$$\overline{s}(t) = \widetilde{r}(t) - \overline{r}(t)$$
$$\overline{s}(t;T) = \widetilde{f}(t;T) - \overline{f}(t;T)$$

Define the LIBOR-discount spread DFs as

$$D_{\overline{s}}(t;T) \equiv \frac{\widetilde{D}(t;T)}{\overline{D}(t;T)}$$

then LIBOR-discount spread curve is:

$$D_{\overline{s}}(t;T)^{-1} = E_t^{\widetilde{\mathbf{F}}(\mathbf{T})}\left[e^{\int_t^T \overline{s}(u)du}\right] = e^{\int_t^T \overline{s}(t;u)du}$$

Forward LIBORs collateralized by foreign cash

OIS-curve based

The foreign-cash collateralized forward LIBOR $\overline{L}(t;T,T+\delta)$ can be calculated as:

$$\overline{L}(t;T,T+\delta) = \frac{1}{\delta}\left(\frac{\widetilde{D}(t;T)}{\widetilde{D}(t;T+\delta)}\exp\left(\int_t^T g(u,T)du\right) - 1\right)$$

where

$$g(t,T) = (b_r(t,T+\delta) - b_r(t,T))\left(\rho_{r,s}(t)b_s(t,T+\delta) - \rho_{r,z}(t)b_z(t,T+\delta)\right) \\ + (b_s(t,T+\delta) - b_s(t,T))\left(b_s(t,T+\delta) - \rho_{s,z}(t)b_z(t,T+\delta)\right)$$

Using the domestic-cash collateralized forward LIBOR,

$$\frac{1+\delta\overline{L}(t;T,T+\delta)}{1+\delta L(t;T,T+\delta)} = \exp\left(-\int_t^T h(u,T)du\right)$$

where

$$\begin{aligned} h(u,T) &= (b_r(u,T+\delta) - b_r(u,T))\,\rho_{r,z}(u)b_z(u,T+\delta) \\ &\quad + (b_s(u,T+\delta) - b_s(u,T))\,\rho_{s,z}(u)b_z(u,T+\delta) \end{aligned}$$

This is the convexity adjustment formula between domestic and foreign-cash collateralized forward LIBORs. Fujii, Takahashi and Shimada (2010) assumes that z is not stochastic. Therefore, in this case there is no convexity adjustment between collateral currencies, that is:

$$\overline{L}(t; T, T+\delta) = L(t; T, T+\delta)$$

There might be a flaw in their argument. They argue that

$$L(t; T, T+\delta) \neq \widetilde{L}(t; T, T+\delta)$$

due to stochastic s. In reality z is more volatile than s.

LIBOR-curve based

We deduce directly from the formula of LIBOR-OIS spread case:

$$\overline{L}(t;T,T+\delta) = \frac{1}{\delta}\left(\frac{\widetilde{D}(t;T)}{\widetilde{D}(t;T+\delta)} E_t^{\widetilde{\mathbf{F}}(\mathbf{T}+\boldsymbol{\delta})}\left[\exp\left(\int_t^T g'(u,T)du\right)\right] - 1\right)$$

where

$$g'(t,T) = \left(b_{\widetilde{r}}(t,T+\delta) - b_{\widetilde{r}}(t,T)\right)\rho_{\widetilde{r},\overline{s}}(t) b_{\overline{s}}(t,T+\delta)$$

therefore,

$$\begin{aligned} & \frac{1+\delta\overline{L}(t;T,T+\delta)}{1+\delta L(t;T,T+\delta)} \\ = \quad & E_t^{\widetilde{\mathbf{F}}(\mathbf{T}+\boldsymbol{\delta})}\left[\exp\left(\int_t^T g'(u,T)du\right)\right] \Big/ E_t^{\widetilde{\mathbf{F}}(\mathbf{T}+\boldsymbol{\delta})}\left[\exp(\int_t^T k'(u,T)du)\right] \end{aligned}$$

In general, there is a convexty adjustment between forward LIBOR collateralized by different currencies even at time 0. It is sufficient to assume that $\rho_{\widetilde{r},\overline{s}}(t) = 0$ and $\rho_{\widetilde{r},s}(t) = 0$ to have no adjustment, i.e.,

$$\overline{L}(t;T,T+\delta) = L(t;T,T+\delta) = \frac{1}{\delta}\left(\frac{\widetilde{D}(t;T)}{\widetilde{D}(t;T+\delta)} - 1\right)$$

Hull-White Model

We do NOT assume the independence between the LIBOR curve and LIBOR-discount spread curves. We apply the Hull-White model to the LIBOR curve and spread curve:

$$\begin{aligned} dX_{\widetilde{r}}(t) &= -\kappa_{\widetilde{r}}(t)X_{\widetilde{r}}(t) + \sigma_{\widetilde{r}}(t)dW_{\widetilde{r}}^{\mathbf{Q}}(t) \\ dX_s(t) &= -\kappa_s(t)X_s(t) + \sigma_s(t)dW_s^{\mathbf{Q}}(t) \\ dW_{\widetilde{r}}(t)dW_s(t) &= \rho_{\widetilde{r},s}(t)dt \end{aligned}$$

In the Hull-White model, $k'(t,T)$ is deterministic function of t and T, i.e.,

$$\frac{1 + \delta L(t; T, T+\delta)}{1 + \delta \widetilde{L}(t; T, T+\delta)} = \exp\left(\int_t^T k'(u, T) du\right)$$

where

$$\begin{aligned}
k'(t, T) &= \left(b_{\widetilde{r}}(t, T+\delta) - b_{\widetilde{r}}(t, T)\right) \rho_{\widetilde{r},s}(t) b_s(t, T+\delta) \\
&= \sigma_{\widetilde{r}}(t) \rho_{\widetilde{r},s}(t) \sigma_s(t) \left(A_{\widetilde{r}}(t, T+\delta) - A_{\widetilde{r}}(t, T)\right) A_s(t, T+\delta) \\
A_i(t, T) &= \int_t^T e^{-\int_t^u \kappa_i(v) dv} du, \ i = \widetilde{r}, \ s
\end{aligned}$$

LIBOR Bond reconstitution formula stays the same:

$$\widetilde{D}(t; T) = \frac{\widetilde{D}(0; T)}{\widetilde{D}(0; t)} \exp\left(-A_{\widetilde{r}}(t, T) X_{\widetilde{r}}(t) - B_{\widetilde{r}}(t, T)\right)$$

where

$$\begin{aligned} B_{\widetilde{r}}(t,T) &= \int_t^T H_{\widetilde{r}}(t,u)du, \\ H_{\widetilde{r}}(t,T) &= \int_0^t \sigma_{\widetilde{r}}(u,T) b_{\widetilde{r}}(u,T)du \\ &= \int_0^t \sigma_{\widetilde{r}}(u)^2 e^{-\int_u^T \lambda_i(v)dv} A(u,T)du \end{aligned}$$

We also have the reconstitution formula for the collateralized forward LIBOR.

$$\begin{aligned} 1+\delta L(t;T,T+\delta) &= (1+\delta L(0;T,T+\delta)) \exp\left(-\int_0^t k'(u,T)du\right) \\ &\quad \times \exp\left[\left(A(t,T+\delta)-A(t,T)\right)X(t)+B(t,T+\delta)-B(t,T)\right] \end{aligned}$$

The reconstitution formula for the LIBOR-discount spread:

$$
\begin{aligned}
D_s(t;T)^{-1} &= \frac{D_s(0;t)}{D_s(0;T)} \exp\left(-A_s(t,T)X_s(t) - B_s(t,T)\right) \\
B_s(t,T) &= \int_t^T H_s(t,u)du \\
H_s(t,T) &= \int_0^t \left[\sigma_{\widetilde{r}}(u;T)b_s(u;T) - \sigma_s(u;T)b_{\widetilde{r}}(u;T)\rho_{\widetilde{r},s}(u) + \sigma_s(u;T)b_s(u;T)\right] dt
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
D(t;T) &= \widetilde{D}(t;T)D_s(t;T)^{-1} \\
&= \frac{D(0;T)}{D(0;t)} \exp\left(-A_{\widetilde{r}}(t,T)X_{\widetilde{r}}(t) - B_{\widetilde{r}}(t,T)\right) \exp\left(-A_s(t,T)X_s(t) - B_s(t,T)\right)
\end{aligned}
$$

Cheyette Model

In the Cheyette model, the bond volatility $b(t;T)$ is stochastic

$$b(t;T) = \sigma(t) \int_t^T e^{-\int_t^u \lambda(v)dv} du$$

due to stochastic $\sigma(t)$.

It is difficult to calculate $E_t^{\widetilde{\mathbf{F}}(\mathbf{T}+\boldsymbol{\delta})} \left[\exp\left(\int_t^T k'(u,T)du \right) \right]$ analytically. Therefore, we restrict the model by the assumption of the indepencence bwtween the LIBOR curve and LIBOR-discount spread curves, i.e., $\rho_{\widetilde{r},s} = 0$ and $\rho_{\widetilde{r},\overline{s}}(t) = 0$.

With this assumption, we have:

$$L(t;T,T+\delta) = \overline{L}(t;T,T+\delta) = \frac{1}{\delta}\left(\frac{\widetilde{D}(t;T)}{\widetilde{D}(t;T+\delta)} - 1\right)$$

However, the LIBOR-discount spread curves s or $\overline{s}$ are still stochastic. The state variables are

$$\begin{cases} dX_{\widetilde{r}}(t) = \left(-\lambda_{\widetilde{r}}(t)X_{\widetilde{r}}(t) + Y_{\widetilde{r}}(t)\right)dt + \sigma_{\widetilde{r}}(t)dW_{\widetilde{r}}^{\mathbf{Q}}(t),\ X_{\widetilde{r}}(0) = 0 \\ dY_{\widetilde{r}}(t) = \left(\sigma_{\widetilde{r}}^2(t) - 2\lambda_{\widetilde{r}}(t)Y_{\widetilde{r}}(t)\right)dt,\ Y_{\widetilde{r}}(0) = 0 \\ dX_s(t) = \left(-\lambda_s(t)X_s(t) + Y_s(t)\right)dt + \sigma_s(t)dW_s^{\mathbf{Q}}(t),\ X_s(0) = 0 \\ dY_s(t) = \left(\sigma_s^2(t) - 2\lambda_s(t)Y_s(t)\right)dt,\ Y_s(0) = 0 \\ dW_{\widetilde{r}}^{\mathbf{Q}}(t)dW_s^{\mathbf{Q}}(t) = 0 \end{cases}$$

where $\sigma_{\widetilde{r}}(t)$ and $\sigma_s(t)$ are stochastic.

The LIBOR bond reconstitution formula stays the same as

$$\widetilde{D}(t;T) = \frac{\widetilde{D}(0;T)}{\widetilde{D}(0,t)} \exp\left(-A_{\widetilde{r}}(t,T)X_{\widetilde{r}}(t) - \frac{1}{2}A_{\widetilde{r}}(t,T)^2 Y_{\widetilde{r}}(t)\right)$$

So the forward LIBOR reconstitution foumula becomes:

$$1 + \delta L(t;T,T+\delta) = (1 + \delta L(0;T,T+\delta)) \\ \times \exp\left[\left(A(t,T+\delta) - A(t,T)\right) X_{\widetilde{r}}(t) + \frac{1}{2}\left(A(t,T+\delta)^2 - A(t,T)^2\right) Y_{\widetilde{r}}(t)\right]$$

And the the inverse of the spread bond formula is:

$$D_s(t;T)^{-1} = \frac{D_s(0;t)}{D_s(0;T)} \exp\left(-A_s(t,T)X_s(t) - \frac{1}{2}A_s(t,T)^2 Y_s(t)\right)$$

Therefore, the discount bond becomes:

$$
\begin{aligned}
D(t;T) &= \widetilde{D}(t;T)\left(D_s(t;T)\right)^{-1} \\
&= \frac{D(0;T)}{D(0;t)} \exp\left(-A_{\widetilde{r}}(t,T)X_{\widetilde{r}}(t) - \frac{1}{2}A_{\widetilde{r}}(t,T)^2 Y_{\widetilde{r}}(t)\right) \\
&\times \exp\left(-A_s(t,T)X_s(t) - \frac{1}{2}A_s(t,T)^2 Y_s(t)\right)
\end{aligned}
$$

LIBOR Market Model

We restrict the model with the assumption of $\rho_{\widetilde{r},s} = 0$ and $\rho_{\widetilde{r},\overline{s}}(t) = 0$

We model canonical risky LIBOR rates $\widetilde{L}_i(t) \equiv \widetilde{L}(t; T_i, T_{i+1}),\ i = 1, \cdots, N$ as

$$d\widetilde{L}_i(t) = \varphi(\widetilde{L}_i(t), t)\lambda_i(t) dW_i^{\widetilde{\mathbf{F}}(\mathbf{T}_{i+1})}(t),\ i = 1, \cdots, N$$

This modeling is the same as in the LIBOR discounting case. Note that $W_i^{\widetilde{\mathbf{F}}(\mathbf{T}_{i+1})}(t) = W_i^{\mathbf{F}(\mathbf{T}_{i+1})}(t)$ and $\widetilde{L}_i(t) = L_i(t)$ when $\rho_{\widetilde{r},s} = 0$ is assumed.

Fujii, Shimada, and Takahashi(2010) modeled the spreads

$$B(t;T_i,T_{i+1}) = L(t;T_i,T_{i+1}) - O(t;T_i,T_{i+1})$$

as $\mathbf{F}(\mathbf{T}_{i+1})$-martingale. This spread modeling is basically a la LIBOR market model. The shortcoming of this approach is that this kind of modeling has too many model parameters. Since the option market for the spread does not exist, it is almost impossible to have all those parameters. Instead of this, we model LIBOR-discount spread as Hull-White model. We can estimate the model parameters,κ_s and σ_s from historical data.

$$\begin{aligned} dX_s(t) &= -\kappa_s(t)X_s(t) + \sigma_s(t)dW_s^Q(t) \\ D_s(t;T)^{-1} &= \frac{D_s(0;t)}{D_s(0;T)} \exp\left(-A_s(t,T)X_s(t) - B(t,T)\right) \end{aligned}$$

where

$$
\begin{aligned}
A_s(t,T) &= \int_t^T e^{-\int_t^u \kappa_s(v)dv} du \\
B_s(t,T) &= \int_t^T H_s(t,u)du \\
H_s(t,T) &= \int_0^t \sigma_s(u,T) b_s(u,T) du
\end{aligned}
$$

Therefore, the future discount bond is, with $t = T_0$ and $T = T_{j+1}$,

$$
\begin{aligned}
D(t;T) &= \widetilde{D}(t;T) D_s(t;T)^{-1} \\
&= \frac{D_s(0;t)}{D_s(0;T)} \exp\left(-A_s(t,T)X_s(t) - B_s(t,T)\right) \prod_{i=1}^{j} \frac{1}{1+\delta \widetilde{L}(t;T_i,T_{i+1})}
\end{aligned}
$$

The future time-t PV of LIBOR cashflow fixed at T_j is

$$\begin{aligned} PV(t) &= \delta L(t; T_j, T_{j+1}) D(t; T) \\ &= \frac{D_s(0; t)}{D_s(0; T)} \exp\left(-A_s(t, T) X_s(t) - B(t, T)\right) \\ &\quad \cdot \delta L(t; T_j, T_{j+1}) \prod_{i=1}^{j} \frac{1}{1 + \delta L(t; T_i, T_{i+1})} \end{aligned}$$

Conclusions

We prefere direct LIBOR curve modeling. To show reasons, we compared OIS-curve based and LIBOR-curve based modeling.

We linked the method of foreign-currency analogy to the stochastic LIBOR-discount spread.

We calculated forward LIBORs collateralized in both domestic and foreign cash.

We showed practical examples of the term structure modeling (Hull White, Cheyette, and BGM).

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